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Linear instability in viscoelastic fluid convection

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Abstract. The onset of convection in a viscoelastic fluid that obeys the Jeffreys model is investigated. Two boundary conditions have been considered separately: free–free and rigid–rigid. The role played by the retardation time, characteristic of the Jeffreys model, is emphasised. The threshold values of the parameters (critical Rayleigh number, critical wavenumber, onset frequency, etc.) for stationary and oscillatory convection are obtained. The frontier between oscillatory and stationary convection is calculated and the possibility to obtain a codimension-two point is discussed.

1. Introduction

Most liquids exhibit viscous and elastic properties for sufficiently small time scales. Studies on light scattering and molecular dynamic simulations show that in order to recover experimental results one must generalise the Newton law relating linearly the extra stress tensor $\mathcal{S}$ and the rate of strain tensor $\mathcal{G}$ (Newtonian fluid) [1]. In particular, this generalisation is necessary in polymeric fluids, glass forming systems, etc. Moreover in these systems the stress tensor enters as an independent variable in the hydrodynamical description. The determination of the dependence of $\mathcal{S} = \mathcal{S}(\mathcal{G})$ is the main object of rheology. The dynamics of the stress tensor give rise to a rich variety of hydrodynamics effects, that cannot appear in the usual Newtonian liquids [2].

In the present study a detailed stability analysis is presented of a layer of fluid with viscoelastic properties heated from below. Usually, the viscoelastic effects are introduced by the Maxwell model, which is the simplest possible model taking into account the stress tensor relaxation. However, as discussed by rheologists this model does not fit the rich variety of viscoelastic effects that can be observed in complex rheological materials [2]. One of the models that shows good agreement with experiments is the so-called Oldroyd model. For a small shear (linear regime) this model reduces to the Jeffreys model that includes the Maxwell fluid as a particular case. Therefore, a more general linear stability analysis of convection in a viscoelastic fluid can be made by taking the Jeffreys model. Moreover, polymeric solutions are, in general, binary mixtures and this can give interesting dynamical effects [3], but here, for the sake of simplicity, we focus on the viscoelastic properties only.

Convection in Jeffreys viscoelastic fluids has three characteristic time scales: a vertical thermal diffusion time τ, characteristic of convective motions, a stress relaxation time

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λ_1 characteristic of the Maxwell viscoelastic model and a retardation time λ_2, that accounts for the corrections of the Jeffreys model. This naturally gives two non-dimensional parameters $\Gamma = \lambda_1/\tau$ and $\Lambda = \lambda_2/\lambda_1$ which enter in the description. The parameter Γ is very small in normal liquids ($\Gamma < 10^{-10}$) but ranges from 0.01 in dilute polymers, to more than 10^4 in polymer melts [4]. The parameter Λ varies in the interval [0, 1]: $\Lambda = 0$ corresponds to a Maxwell fluid, while with $\Lambda = 1$ the Newtonian fluid is recovered.

Some results on the linear stability problem of convection in viscoelastic fluid have been treated by some authors [5]. Besides the usual stationary convection these authors have found that an oscillatory state can also be obtained for certain values of the fluid parameters. This oscillatory state is the consequence of two competing timescales. It can be observed experimentally in the form of travelling or standing waves. A complete study of these solutions and their stability requires a non-linear analysis. (For particular cases some weakly non-linear studies have been made [6]; however, these studies do not allow a complete discussion of this problem.) Therefore, the present study can be viewed as a first step (linear theory) towards a non-linear stability analysis of convection in viscoelastic fluids.

In most cases the stability problem of convection in a viscoelastic fluid have been performed theoretically under unrealistic free–free boundary conditions (BC) for which analytical solutions are possible. But, as stressed in similar problems of convection (binary mixtures) [7], BC can introduce qualitative changes in the dynamics of the system. Therefore, we recast briefly the main results for free–free BC, before calculating stability with rigid–rigid BC. In the last case the solutions must be obtained numerically, but the results can be compared with experiments. One of the purposes of the present study is to determine the role of these BC on convection in a viscoelastic fluid.

Oscillatory and stationary branches match at a given point where overstable motions lose their stability in favour of stationary convection. It is well known from general stability theory of dynamical systems that some interesting phenomena can be observed at those singular points [8]. Similar situations in convection in binary mixtures have been studied with some detail from both theoretical and experimental points of view, in particular the existence and observability of oscillatory modes and of codimension-two (CT) points [9–11]. The competition between oscillatory and stationary modes of convection in Maxwell viscoelastic fluids has been discussed recently ([4] and Zielinska and co-workers [6]). However, this competition is found for wavenumbers and frequencies which seem very unrealistic and only for free–free BC. Another purpose of the present paper is to examine the influence of the parameter Λ of the Jeffreys model on the values of the wavenumbers and frequencies of stationary and oscillatory convection under different BC.

The paper is organised as follows: in § 2 we discuss the main evolution equations and the constitutive equation of the Jeffreys viscoelastic model. In § 3 we present the solutions and numerical results, separately in two sub-sections. In the first one we recast briefly the main results for free–free BC. After discussing the numerical recipe to solve the eigenvalue problem, the results for rigid–rigid boundary conditions are presented in § 3.2. We focus on the competition between oscillatory and stationary convection for different values of the parameter Λ and the Prandtl number. The similarities and differences of convective solutions in these two cases are also discussed in this sub-section. This is followed by a general discussion and some conclusions in § 4.

2. Equations of convection in viscoelastic fluids

We consider a shallow layer of an incompressible fluid of depth d and of infinite horizontal extent. The fluid is heated from below and remains at rest until a critical temperature gradient is reached. After assuming the Oberbeck–Boussinesq (OB) approximation the basic equations for this fluid can be written as

$$\nabla \cdot \boldsymbol{v} = 0 \tag{1}$$

$$\rho_0[\partial \boldsymbol{v}/\partial t + \boldsymbol{v} \cdot \nabla \boldsymbol{v}] = -\nabla p + \nabla \mathcal{J} + \rho \boldsymbol{g} \tag{2}$$

$$\partial T/\partial t + \boldsymbol{v} \cdot \nabla T = \kappa \nabla^2 T \tag{3}$$

where $\boldsymbol{v}$ is the velocity field, p is the pressure, T the temperature, $\mathcal{J}$ the extra stress tensor and $\boldsymbol{g}$ is the acceleration due to gravity. As usual in the OB approximation one assumes that the fluid is incompressible except in the term of the gravitational force where $\rho = \rho_0[1 - \alpha(T - T_0)]$ and the thermal expansion coefficient α and the thermal diffusivity κ are constants.

To solve these equations it is necessary to assume a constitutive relation between the extra pressure tensor $\mathcal{J}$ and the velocity gradient tensor $\nabla \boldsymbol{v}$. One of the more general models to study complex rheological behaviour is the Oldroyd model with eight parameters [1]. As in the present paper we only intend to study linear properties we restrict this model to its linearised version, which takes the name of Jeffreys model, for which the constitutive relation writes as

$$\mathcal{J} + \lambda_1\, \partial \mathcal{J}/\partial t = \mu(\mathcal{G} + \lambda_2\, \partial \mathcal{G}/\partial t) \tag{4}$$

where $\mathcal{G}$ is the symmetric part of the strain tensor $\mathcal{G} = [\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T]$, μ the shear viscosity coefficient, assumed to be constant in the OB approximation, and λ_1 and λ_2 the relaxation time and the retardation time respectively (two quantities that are very small in normal fluids). Notice that $\lambda_1 \geqslant \lambda_2$ and that when $\lambda_2 = 0$ we recover the Maxwell viscoelastic model while for $\lambda_1 = \lambda_2$ the model reduces to the Newtonian fluid.

The stationary solution of the system of equations (1)–(4) is simply $\boldsymbol{v} = \boldsymbol{0}, \mathcal{J} = \boldsymbol{0}, T = T_1 - \beta z$ where β is the adverse temperature gradient. The linear equations for the perturbations of this conduction state, after scaling the variables using d, d^2/κ, κ/d, $\mu\kappa/d^2$ and βd as references for the length, time, velocity, extra stress tensor and temperature, respectively, take the form

$$(\partial/\partial t - \nabla)T = w \tag{5}$$

$$\begin{aligned}[(1 + \Lambda\Gamma\, \partial/\partial t)\nabla^2 - (1 + \Gamma\, \partial/\partial t)P^{-1}\, \partial/\partial t]\nabla^2 w \\ = -(1 + \Gamma\, \partial/\partial t)R\nabla_1^2 T\end{aligned} \tag{6}$$

where $P = \mu/\rho_0\kappa$ is the Prandtl number, $R = \rho_0 g\alpha\beta d^4/\mu\kappa$ is the Rayleigh number, $\Gamma = \lambda_1\kappa/d^2$ the non-dimensional relaxation time, $\Lambda = \lambda_2/\lambda_1$ the retardation ratio and $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

We study equations (5) and (6) with the following set of boundary conditions (BC)

(i) $w = \mathrm{D}^2 w = T = 0 \quad \text{at } z = 0, 1$ (7*a*)

(free–free and conducting

(ii) $w = \mathrm{D}w = T = 0 \quad \text{at } z = 0, 1$ (7*b*)

(rigid–rigid and conducting)

where $\mathrm{D} = \mathrm{d}/\mathrm{d}z$. As usual the solutions are developed in normal modes

$$\begin{bmatrix} w \\ T \end{bmatrix} = \begin{bmatrix} W(z) \\ \theta(z) \end{bmatrix} \exp\{\mathrm{i}(k_x x + k_y y) + \sigma t\} \tag{8}$$

where $W(z)$ and $\theta(z)$ are the amplitudes of velocity and the temperature perturbations, k_x and k_y are the components of the wavenumber in the horizontal plane and $\sigma = \sigma_r + \mathrm{i}\omega$

the growth factor. Equations (5) and (6) then become

$$[\sigma - (\mathrm{D}^2 - k^2)]\theta = W \tag{9}$$

$$[(1 + \Gamma\Lambda\sigma)(\mathrm{D}^2 - k^2) - (1 + \Gamma\sigma)P^{-1}\sigma](\mathrm{D}^2 - k^2)W = (1 + \Gamma\sigma)k^2 R\theta. \tag{10}$$

Equations (9) and (10) and the BC (7) constitute an eigenvalue system for the linear stability problem. The eigenvalue problem has been solved by Vest and Arpaci [5] for the Maxwell fluid. Green [5], in an early article, analysed the Jeffreys model but only with free–free BC and without discussing conditions for coexisting modes.

3. Solutions and numerical results

The system loses stability when the real part of the growth rate is greater than zero $\sigma_r \geqslant 0$. The marginal case $\sigma_r = 0$ separates stable and unstable regions. Then, two situations can appear: stationary convection, when the imaginary part also vanishes and oscillatory convection when this part is non-zero.

We present the results of these eigenvalue problems for the two kinds of BC separately because the method of solution is different in each case. In both cases the curves of marginal stability, giving the dependence of R on k for stationary and oscillatory convection, are obtained. The minima of these curves determine the corresponding critical values $R^c_{o,s}$ and $k^c_{o,s}$. (Here the subscripts o and s indicate oscillatory and stationary, respectively.) The values of these critical parameters for overstable and stationary convection are different.

Then, if the system can choose its wavenumber freely (unbounded system) convection starts with that mode which has the minimum R^c. As R^c_s does not depend on the viscoelastic parameters we focus on overstable convection in the following. This is possible when $R^c_o \leqslant R^c_s$. More specifically we will study the particular case when these two thresholds are equal, i.e., the frontier between overstable and stationary motions. We explore the regions in the parameters space in which these two modes can coexist, first for BC (i) (equation (7*a*)) where an analytical solution is possible and second for the realistic BC (ii) (equation (7*b*)).

Another interesting situation can be found, when one imposes the wavenumber of convective motions externally (by means of lateral walls or by some external forcing). Then one can adjust the wavenumber to the value for which stationary and oscillatory curves meet and, therefore, $\omega \to 0$. At this CT-point both marginally stable modes can coexist simultaneously and some interesting dynamical phenomena can occur [8]. However, this CT point is hardly observable under usual experimental conditions.

3.1. Free–free boundary conditions

Equations (9) and (10) coincide with the stationary case for Newtonian fluids. As a consequence the marginal curves for stationary convection are also identical in these two cases. These correspond to $R_s = (\pi^2 + k^2)^3/k^2$, giving the critical values for stationary convection $R^c_s = 27\pi^4/4$ for $k^c_s = \pi/\sqrt{2}$. For oscillatory convection the marginal curve is given by Rosenblat [6]

$$R_o = R_s - (\pi^2 + k^2)\omega^2\{[(\pi^2 + k^2)\Gamma + 1]P^{-1} + (\pi^2 + k^2)\Gamma\Lambda\}/k^2 \tag{11}$$

where ω corresponds to the imaginary part of the growth parameter σ, given by

$$\omega^2 = [(\pi^2 + k^2)\Gamma(1 - \Lambda) - 1 - P^{-1}]/[\Gamma^2(P^{-1} + \Lambda)]. \tag{12}$$

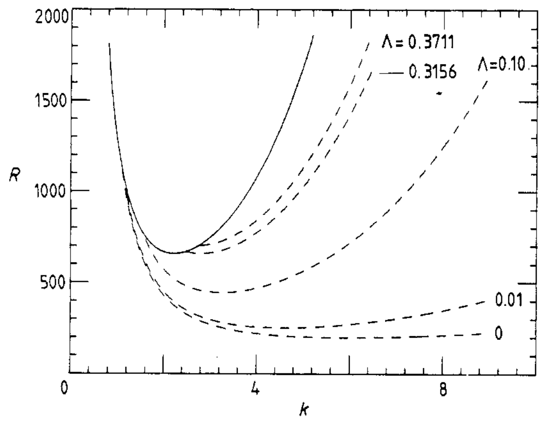

Figure 1. Marginal curves for free–free BC with $P = 10, \Gamma_1 = 0.1$. The full curve refers to stationary convection, the broken curves to overstability.

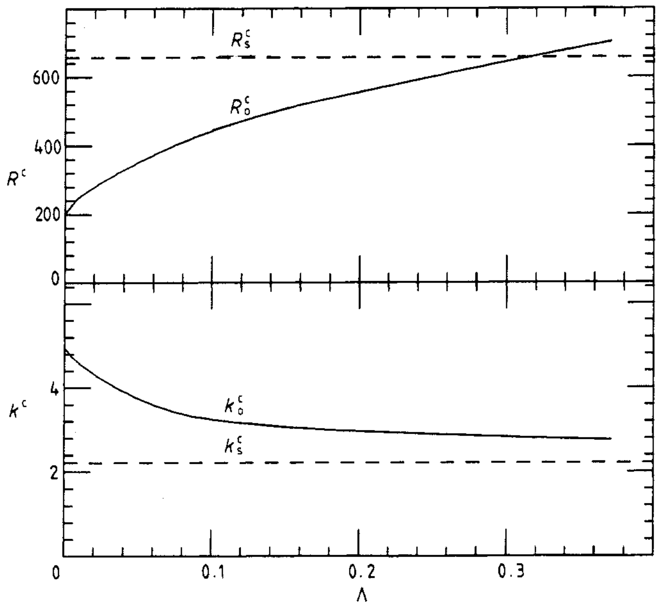

Figure 2. Critical Rayleigh number R^c and critical wavenumber k^c for stationary and oscillatory instabilities, as a function of Λ for $\Gamma = 0.1$, $P = 10.0$ (free–free BC).

As ω is real, this curve gives overstability only for $\omega^2 > 0$. The corresponding critical values R_o^c and k_o^c for oscillatory convection may be obtained numerically. (From these values and (12) the corresponding critical frequency ω^c is easily obtained.)

In figure 1 we illustrate some typical marginal curves (R as a function of k). The full curve, which is independent of P and Γ, concerns stationary convection. Broken curves correspond to overstability for several values of Λ, with $P = 10.0$ and $\Gamma = 0.1$. For $\Lambda = 0$ (Maxwell model) we recover the values obtained by Vest and Arpaci [5] and Sokolov and Tanner [5]. Figure 2 shows the critical wavenumbers and corresponding critical Rayleigh numbers, as functions of the parameter Λ for $\Gamma = 0.1$ and $P = 10.0$. The critical wavenumber k_o^c decreases and the critical Rayleigh number R_o^c increases with increasing Λ.

For $\Lambda_f = 0.3156$ the critical Rayleigh numbers coincide (the values that characterise these frontier points will be labelled with the subscript f in the following), and the rest of the critical values are $k_{s,f}^c = 2.221$, $k_{o,f}^c = 2.810$ and $\omega_f^c = 5.279$. This is the point

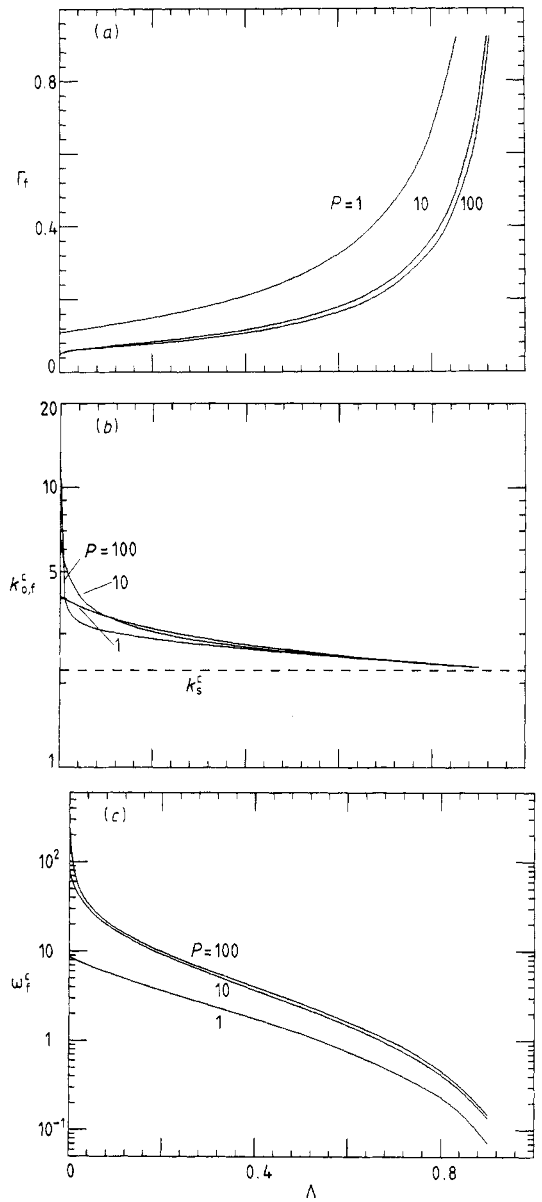

Figure 3. Dependence of (a) the relaxation time Γ_f, (b) the critical wavenumber $k^c_{o,f}$ and (c) the critical frequency ω^c_f at the crossover between oscillatory and stationary convection, as a function of the Λ for various values of P (free–free BC).

where the lowest threshold changes from oscillatory to stationary instabilities. For $\Lambda > \Lambda_f$ overstable motions cannot appear spontaneously in a system of infinite horizontal extent. However, overstable motions are still possible in a system with a fixed wavenumber. In this situation CT points can be reached. In particular, the quadratic minimum in the curve $R_o(k)$ disappear for $\Lambda(\text{CT}) = 0.3711$, $k^c(\text{CT}) = 2.761$, $R^c(\text{CT}) = 702.1$ and $\omega^c = 0$ (for $P = 10.0$ and $\Gamma = 0.1$) leading to a degenerate CT point.

We present now the results on the influence of the parameter Λ and P on that frontier between oscillatory and stationary convection. Figure 3 gathers the main results. In figure 3(a) we show the dependence of Γ_f as a function of the parameter Λ. In the region above these curves the system is unstable under oscillatory convection. (Stationary motions are the unstable modes below these curves.) For a fixed Λ, Γ_f decreases with increasing P. This means that the higher the viscous effects, the lower the relaxation time necessary to start overstability for a fixed R. In the case of $\Lambda = 0$ (Maxwell fluid) and $P \rightarrow \infty$ the minimum value $\Gamma_f = 0.039$ is obtained (Zielinska and co-workers [6]). The results of the Newton fluid is recovered in the limit of $\Lambda = 1$ independently of the value of Γ. This is the reason for the divergence of Γ_f in that limit.

The values of the critical wavenumber for oscillatory instability in the frontier $k^c_{o,f}$ as a function of Λ is quoted in figure 3(*b*). Notice that in the limit $\Lambda = 0$ (Maxwell model) the values of $k^c_{o,f}$ are greater than $k^c_{s,f}$, and the difference between them increases with P. (For example for $P = 100$ and $\Lambda = 0$, $k^c_{o,f}/k^c_{s,f} = 5.6$ and $\omega^c_f = 579.8$, but that ratio decreases to 1.8 and the frequency to $\omega^c_f = 81.7$ for $\Lambda = 0.01$.) With increasing Λ, $k^c_{o,f}$ and ω^c_f decrease rapidly until in the limit $\Lambda \rightarrow 1$ (Newtonian fluid), one recovers $k^c_o \rightarrow k^c_s$ independently of the value of P.

The oscillation frequency ω^c_f as a function of Λ is shown in figure 3(*c*). This frequency ω^c_f is very large in the limit $\Lambda \rightarrow 0$, while it tends to zero when $\Lambda \rightarrow 1$, because overstability disappears in this limit. From these results we remark that in the pure Maxwell model the critical wavenumber and the oscillation frequency increase monotonically when P increases, reaching very unrealistic values. However, with a small retardation time or, equivalently, a small Λ, we recover more reasonable values for the critical parameters.

3.2. *Rigid–rigid boundary conditions*

We examine in this sub-section the consequences of the rigid–rigid BC in convection in viscoelastic fluids. In principle, this drastic change in the BC, compared with the free–free case, could lead to qualitative changes in the dynamics of the system. Moreover the results with these realistic BC could be compared with experiments. We start from (9) and (10) with the BC (ii) (7*b*), which constitute an eigenvalue problem whose exact solution entails a rather involved numerical procedure [5]. Here we take a useful numerical method introduced by Chock and Schechter [12] that converts the BC problem into an initial value problem. This method has been widely used in hydrodynamical instability problems [13]. The main steps of this method are explained in the following.

First we take as variables $u_1 = w$, $u_2 = Dw$, $u_3 = D^2w$, $u_4 = D^3w$, $u_5 = T$, $u_6 = DT$, which obey a system of differential equations in the form

$$Du_i = M_{ij}(P, R, k, \sigma, \Gamma, \Lambda)u_j \tag{13}$$

where the components M_{ij} of matrix $\mathbf{M}$ are straightforwardly obtained from equations (9) and (10). The natural BC (7*b*) are rewritten for these variables as

$$u_1(0) = u_2(0) = u_5(0) = 0 \tag{14a}$$

$$u_1(1) = u_2(1) = u_5(1) = 0. \tag{14b}$$

This system admits a set of six linearly independent solutions u^j_i where the superscript j is the index for these solutions j ($j = 1, \ldots, 6$) for each variable u_i. As usual, we choose the initial values $u^j_i(0) = \delta_{ij}$. The BC (14) imply the following relation

$$\det(u^j_i(1)) = 0 \qquad i = 1, 2, 5 \qquad j = 3, 4, 6. \tag{15}$$

From this relation and the general solutions of (13) one obtains the marginal curves. The advantage of this numerical method is its quick convergence, although some problems can appear for very high values of the fluid parameters [13].

The stability curves for stationary and oscillatory convection are shown in figure 4,

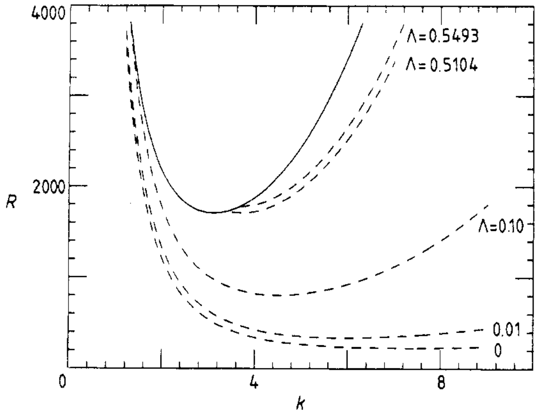

Figure 4. Marginal curves for rigid–rigid BC. The full curve refers to stationary convection, the broken curves to overstability.

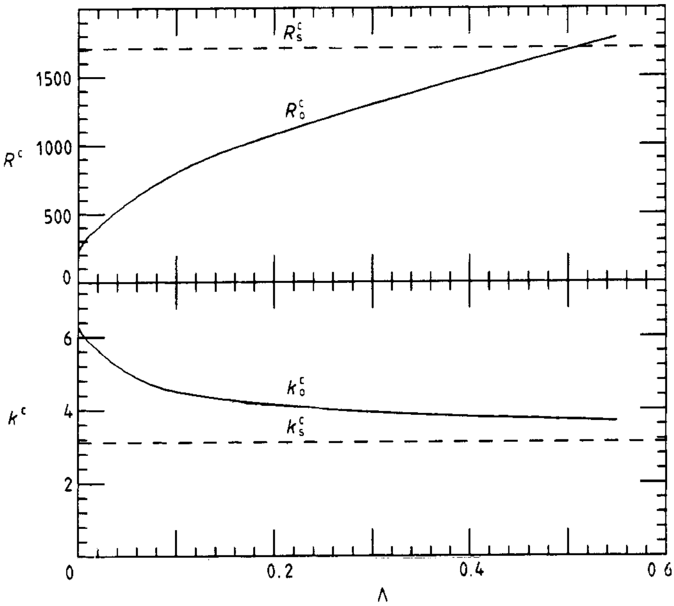

Figure 5. Critical Rayleigh number and critical wavenumber for stationary and oscillatory instabilities, as a function Λ for $\Gamma = 0.1$, $P = 10.0$ (rigid–rigid BC).

for different values of the ratio Λ. In general, the thresholds are higher for BC (ii) than for BC (i). Figure 5 shows the critical Rayleigh number and the critical wavenumber for stationary and oscillatory instabilities, as a function of Λ for $P = 10$ and $\Gamma = 0.1$. We obtain that $R^c_s = R^c_o$ for $\Lambda_f = 0.5104$. In this case also the wavenumbers are unequal at this frontier point $k^c_s = 3.116$, $k^c_{o,f} = 3.716$, $\omega^c_f = 4.276$. The degenerate CT point is obtained for $\Lambda(\text{CT}) = 0.5493$, $k^c(\text{CT}) = 3.680$ and $R^c(\text{CT}) = 1780.3$ with $\omega^c = 0$. The influence of the ratio Λ on the convection is also qualitatively similar to that obtained for BC (i).

Figure 6 shows the critical values of the parameters Γ_f, $k^c_{o,f}$ and ω^c_f at the frontier between oscillatory and stationary instability. These results are qualitatively the same as those in figure 3 for BC (i), the only difference being that Γ_f (i) $>$ Γ_f (ii), $k^c_{o,f}$ (i) $<$ $k^c_{o,f}$ (ii) and ω^c_f (i) $<\omega^c_f$ (ii). Therefore the main result in this subsection is that the BC for the velocity field introduce only quantitative, but not qualitative, changes in the convective thresholds.

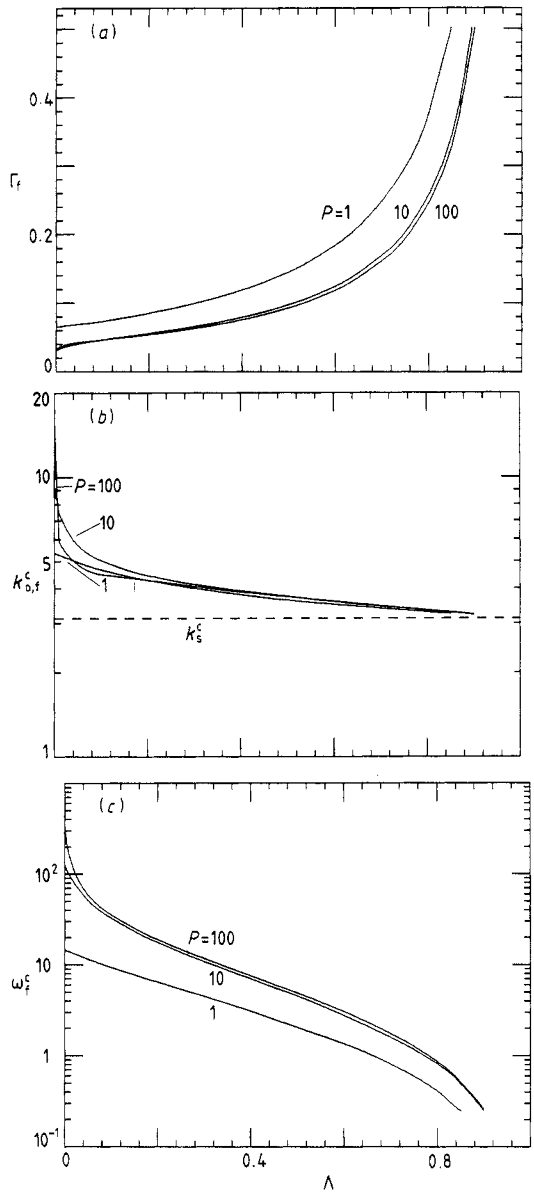

Figure 6. Dependence of (*a*) the relaxation time Γ_f, (*b*) the critical wavenumber $k^c_{o,f}$ and (*c*) the critical frequency ω^c_f at the crossover between oscillatory and stationary convection, as a function of Λ for various values of P (rigid–rigid BC).

4. Summary and conclusions

We have solved the linear stability theory of a viscoelastic fluid obeying the Jeffreys model. This model accounts for the elasticity of the fluid by means of a relaxation time and a retardation time for stresses. It includes the Maxwell model as a particular case. As stressed by many authors, the Jeffreys model is one of the most suitable linear models to compare with experiments. We find that oscillatory convection exists between the two limiting cases of Maxwellian and Newtonian fluids. Free–free boundary conditions (i) and the more realistic rigid–rigid ones (ii) have been treated separately.

The existence of codimension-two (CT) point and the values of the critical parameters in these points have been also determined. Although possible, these points are hardly observable under normal experimental convective conditions.

The dependence of the critical wavenumber $k_{0,f}^{c}$ and frequency ω_f^c, and the value of the relaxation time Γ_f at the frontier where the lowest threshold changes from oscillatory to stationary convection have been determined for different values of the retardation ratio Λ and the Prandtl number P. The present calculations show that the Jeffreys model allows one to correct some of the limitations of the Maxwell model for convection. In the limit of a Maxwellian fluid ($\Lambda = 0$) these critical values are very unrealistic for high P. However, with the Jeffreys model, even for small retardation ratios Λ the values of $k_{0,f}^{c}$ and ω_f at this frontier enter into the observable range.

The differences on the critical parameters for the two kinds of boundary conditions (BC) examined here are only quantitative and, as expected, the system is more stable in the rigid–rigid situation.

Acknowledgments

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